Embedding Boolean ample monoids as full submonoids of Boolean inverse monoids

by

Mark V. Lawson
Heriot-Watt University

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1. Classical Stone duality

Boolean space =

compact, Hausdorff 0-dimensional.

Theorem (Stone, 1937)

The category of Boolean algebras is dually equivalent to the category of Boolean spaces.

Method

- Boolean algebra → Boolean space

  use prime filters (= ultrafilters)

- Boolean space → Boolean algebra

  use clopen (= closed and open) sets

This can be generalized . . .

2. Non-commutative Stone duality

Idea

- Replace Boolean algebras by monoids having a "Boolean character".

- Replace Boolean spaces by topological categories generalizing Boolean spaces.

A <u>topological category</u> is a category equipped with a topology in which all the maps $\underline{d}$, $\underline{r}$, $\underline{m}$ are continuous.

A topological category is <u>étale</u> if $\underline{d}$ and $\underline{r}$ are local homeomorphisms.

<u>Why étale?</u>

The open subsets form a monoid.

Thus étale topological categories have a monoid "alter ego"!

In a restriction monoid, define the <u>compatibility relation</u> $\sim$ by

$a \sim b$ iff $ab^* = ba^*$ and $a^+ b = b^+ a$.

Define the <u>natural partial order</u> $\leqslant$ by $a \leqslant b$ iff $a = ba^*$ iff $a = a^+ b$.

If $a, b \leqslant c$ then $a \sim b$.

Thus, being compatible is a necessary condition for a pair of elements to have a join.

An ample monoid is said to be <u>Boolean</u> if

(1) Each pair of compatible elements has a join.

(2) Multiplication distributes over such joins.

(3) The idempotents form a Boolean algebra w.r.t. the natural partial order.

(4) The zero of the Boolean algebra is the zero of the monoid.

A Boolean category is an étale topological category whose space of identities is a Boolean space.

<u>Theorem</u> (Kudryavtseva – Lawson, 2017)

(i) Associated with every Boolean ample monoid S is a cancellative Boolean category $\underline{C}(S)$.

(ii) Associated with every cancellative Boolean category C is an ample monoid $\underline{KB}(C)$.

(iii) If S is a Boolean ample monoid then $S \cong \underline{KB}(\underline{C}(S))$

(iv) If C is a cancellative Boolean category then

$C \cong \underline{C}(\underline{KB}(C))$.

- $\underline{C}(S)$ is obtained from S by taking the prime filters in S.

- $\underline{KB}(C)$ is obtained from C by taking the compact-open local bisections in C.

- If we replace Boolean ample monoids by Boolean inverse monoids then the category $\underline{C}(S)$ is a groupoid.

3. Full embedding theorem

Idea

Boolean inverse monoids : groupoids

as

Boolean ample monoids : cancellative categories

The question is:

under what circumstances can a cancellative category be "nicely" embedded in a groupoid.

A cancellative category C is said to be <u>right reversible</u> if for all $a, b \in C$ s.t. $\underline{d}(a) = \underline{d}(b)$ we have that $Ca \cap Cb \neq \emptyset$.

<u>Theorem</u> (Gabriel - Zisman, 1967)

Let C be a right reversible cancellative category. Then there exists an embedding $\iota : C \to G$ into a groupoid G s.t.

(1) $\iota(C)^{-1}\iota(C) = G$.

(2) $\iota(C)$ is a wide subcategory of G.

We call G a <u>groupoid of fractions</u> of C.

Key Result

Theorem (Lawson, 2026)

Let C be a cancellative Boolean category which is right reversible. Then its groupoid of fractions is a Boolean groupoid.

Remark

We have added topology to the result of Gabriel-Zisman.

Let S be a Boolean ample monoid. Let F be a prime filter of $\underline{E}(S)$. We write

$x \leq_F y$ if $x^*, y^* \in F$ and $x \leq y$.

We say that S satisfies (C) if the following hold:

for any prime filter $F \subseteq \underline{E}(S)$ and $a, b \in S$ s.t. $a^* b^* \in F$

there exist $c, d \in S$ s.t.

(1) $\exists\, a_1 \leq_F a$ s.t. $a_1^+ \leq c^*$

$\exists\, b_1 \leq_F b$ s.t. $b_1^+ \leq d^*$

(2) $\exists\, e \in F$ s.t. $cae = dbe$

<u>Proposition</u> Let S be a Boolean ample monoid satisfying (C). Then for all $a, b \in S$ s.t. $a^* b^* \neq 0$ we have that $Sa \cap Sb \neq \{0\}$.

<u>Theorem</u> Let S be a Boolean ample monoid. Then (C) holds iff $\underline{C}(S)$ is right reversible.

Main theorem

Let S be a Boolean ample monoid satisfying (C).
Then S can be embedded as a full submonoid of a Boolean inverse monoid T in such a way that every non-zero element of T is a finite join of elements of the form $a^{-1}b$ where $a, b \in S$.

<u>Proof (Sketch)</u>

Let S be a Boolean ample monoid satisfying (c).

Then $\underline{C}(S)$ is a cancellative Boolean category which is right reversible. Thus $\underline{C}(S)$ has a groupoid of fractions G.

In fact, G is a Boolean groupoid.

By non-commutative Stone duality S can be fully embedded in a Boolean inverse monoid T.